EXTREMELY CHARGED STATIC DUST DISTRIBUTIONS IN GENERAL RELATIVITY

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Abstract

Conformo static charged dust distributions are investigated in the framework of general relativity. Einstein's equations reduce to a non-linear version of Poisson's equation and Maxwell's equations imply the equality of the charge and mass densities. An interior solution to the extreme Reissner-Nordström metric is given. Dust distributions concentrated on regular surfaces are discussed and a complete solution is given for a spherical thin shell.

1 Introduction

Let M be a four dimensional spacetime with the metric

$$g_{\mu\nu} = f^{-1}\,\eta_{1\,\mu\nu} - u_\mu\,u_\nu \tag{1}$$

where $\eta_{1\,\mu\nu} =$ diag $(0,1,1,1)$ and $u_\mu = \sqrt{f}\,\delta^0_\mu$. Here Latin letters represent the space indices and δ_{ij} is the three dimensional Kronecker delta. In this work we shall use the same convention as in [1]. The only difference is that we use Greek letters for four dimensional indices. Here M is static. The inverse metric is given by

$$g^{\mu\nu} = f\,\eta_2^{\mu\nu} - u^\mu\,u^\nu \tag{2}$$

where $\eta_{2\,\mu\nu} =$ diag $(0,1,1,1)$, $u^\mu = g^{\mu\nu}\,u_\nu = -\frac{1}{\sqrt{f}}\,\delta^\mu_0$. Here u^μ is a time-like four vector, $u^\mu\,u_\mu = -1$.

The Maxwell antisymmetric tensor and the corresponding energy momentum tensor are respectively given by

$$F_{\mu\nu} = \nabla_\nu\,A_\mu - \nabla_\mu\,A_\nu \tag{3}$$

$$M_{\mu\nu} = \frac{1}{4\pi}\,(F_{\mu\alpha}\,F^\alpha_\nu - \frac{1}{4}\,F^2\,g_{\mu\nu}) \tag{4}$$

where $F^2 = F^{\mu\nu}\,F_{\mu\nu}$. The current vector j^μ is defined as

$$\nabla_\nu\,F^{\mu\nu} = 4\pi\,j^\mu \tag{5}$$

The Einstein field equations for a charged dust distribution are given by

$$G_{\mu\nu} = 8\,\pi\,T_{\mu\nu} = 8\pi\,M_{\mu\nu} + (8\,\pi\,\rho)\,u_\mu\,u_\nu \tag{6}$$

where ρ is the energy density of the charged dust distribution and the four velocity of the dust is the same vector u^μ appearing in the metric tensor. Very recently [6] we invetstigated the above filed equations with metric given in (1). We find that $j^\mu = \rho_e\,u^\mu$, where ρ_e is the charge density of the dust distribution. Let λ be a real function depending on the space coordinates. It turns out that $A_i = 0$ and

$$f = \frac{1}{\lambda^2}, \quad A_0 = \frac{k}{\lambda} \tag{7}$$

where $k = \pm 1$. Then the field equations reduce simply to the following equations.

$$\nabla^2 \lambda + 4\,\pi\,\rho\,\lambda^3 = 0 \tag{8}$$

$$\rho_e = k\,\rho \tag{9}$$

where ∇^2 denotes the three dimensional Laplace operator in Cartesian flat coordinates. These equations represent the Einstein and Maxwell equations respectively. In particular the first equation (8) is a generalization of the Poisson's potential equation in Newtonian gravity. When ρ vanishes , the space-time metric describes the Majumdar-Papapetrou space-times [2] , [3] , [4] , [5]. For the case $\rho \neq 0$, the reduced form of the field equations (8) were given quite recently [6] (see also [8]).

2 Charged dust clouds

In the Newtonian approximation $\lambda = 1 + V$, Eq.(8) reduces to the Poisson equation, $\nabla^2 V + 4\pi\,\rho = 0$. Hence for any physical mass density ρ of the dust distribution we solve the equation (8) to find the function λ. This determines the space-time metric completely. As an example for a constant mass density $\rho = \rho_0 > 0$ we find that

$$\lambda = \frac{a}{2\,\sqrt{\pi\,\rho_0}}\,cn(l_i\,x^i) \tag{10}$$

Here l_i is a constant three vector ,$a^2 = l_i l^i$ and cn is one of the Jacobi elliptic function with modulus square equals $\frac{1}{2}$. This is a model universe which is filled by a (extreme) charged dust with a constant mass density.

3 Interior solutions

In an asymptotically flat space-time , the function λ asymptotically obeys the boundary condition $\lambda \to \lambda_0$ (a constant). In this case we can establish

the equality of mass and charge $e = \pm m_0$, where $m_0 = \int \rho \sqrt{-g}\, d^3x$. For physical considerations our extended MP space-times may be divided into inner and outer regions. The interior and outer regions are defined as the regions where $\rho_i > 0$ and $\rho = 0$ respectively. Here $i = 1, 2, ..., N$, where N represents the number of regions. The gravitational fields of the outer regions are described by any solution of the Laplace equation $\nabla^2 \lambda = 0$, for instance by the MP metrics. As an example the extreme Reissner-Nordström (RN) metric (for $r > R_0$), $\lambda = \lambda_0 + \dfrac{\lambda_1}{r}$ may be matched to a metric with

$$\lambda = a \frac{\sin(b\, r)}{r}, \;\; r < R_0 \tag{11}$$

describing the gravitational field of an inner region filled by a spherically symmetric charged dust distribution with a mass density

$$\rho = \rho(0)\, [\frac{b\, r}{\sin(b\, r)}]^2 \tag{12}$$

Here $\rho(0) = \frac{1}{4\,\pi\, a^2}$, $r^2 = x_i\, x^i$, a and b are constants to be determined in terms of the radius R_0 of the boundary and total mass m_0 (or in terms of $\rho(0)$). The boundary condition , when reduced on the function λ on the surface $r = R$ it must satisfy both $\lambda_{out} = \lambda_{in}$ and $\lambda'_{out} = \lambda'_{in}$. Here prime denotes differentiation with respect r. They lead to [7]

$$\begin{aligned} b\, R_0 &= \sqrt{\frac{3 m_0}{R_0}}, & (13)\\ \lambda_0 &= ab \cos(bR_0), & (14)\\ \lambda_1 &= a\,(\sin(bR_0) - bR_0\, \cos(bR_0)) & (15) \end{aligned}$$

When the coordinates transformed to the Schwarzschild coordinates .i.e., $\lambda_0\, r + \lambda_1 \longrightarrow r$ then the line element becomes

$$ds^2 = -\frac{1}{\lambda_0^2}(1 - \frac{\lambda_1}{r})^2\, dt^2 + \frac{dr^2}{(1 - \frac{\lambda_1}{r})^2} + r^2\, d\Omega^2. \tag{16}$$

hence λ_1 is the mass in the Newtonian approximation then $\lambda_1 = m_0$.
In this way one may eliminate the singularities of the outer solutions by matching them to an inner solution with a physical mass density.

For the mass density $\rho = \frac{b^2}{4\pi\,\lambda^2}$ in general , we may have the complete solution. Here b is a nonzero constant which is related to m_0 by the relation $b^2\,R_0^3 = 3m_0$ and we find that

$$\lambda = \sum_{l,m} a_{l,m}\, j_l(b\,r)\, Y_{l,m}(\theta,\phi) \tag{17}$$

where $j_l(b\,r)$ are the spherical Bessel functions which are given by

$$j_l(x) = (-x)^l\,(\frac{1}{x}\,\frac{d}{dx})^l\,(\frac{\sin x}{x}) \tag{18}$$

and $Y_{l,m}$ are the spherical harmonics. The constants $a_{l,m}$ are determined when this solution is matched to an outer solution with $\nabla^2\lambda = 0$. The interior solution given above for the extreme RN metric with density (12) corresponds to $l = 0$.

4 Point particle solutions

Newtonian gravitation is governed by the Poisson type of linear equation. Gravitational fields of spherical objects in the exterior regions may also be identified as the gravitational fields of masses located at a discrete points (point particles located at centers of the spheres) in space (R^3). The solution of the Poisson equation with N point singularities may be given by

$$\lambda = 1 + \sum_{i=1}^{N} \frac{m_i}{r_i}, \tag{19}$$

$$r_i = [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{\frac{1}{2}} \tag{20}$$

where N point particles with masses m_i are located at the points (x^i, y^i, z^i) with $i = 1, 2, , , N$. The same solution given above may also describe the exterior solution of the N spherical objects (with nonempty interiors) with total masses m_i and radii R_i. The interior gravitational fields of such spherical objects can be determined when the mass densities ρ_i are given. The essential point here is that the limit $R_i \to 0$ is allowed. This means that the dust distribution is replaced by a distribution concentrated at the points (x^i, y^i, z^i) . Namely in this limit mass densities behave as the Dirac delta functions,

$$\rho \to \sum_{i=1}^{N} m_i \, \delta(x - x_i) \, \delta(y - y_i) \, \delta(z - z_i).$$

This is consistent with the Poisson equation $\nabla^2 \lambda + 4\pi \, \rho \, = 0$, because $\frac{1}{r_i}$ in the solution (19) is the Green's function, i.e.,

$$\nabla^2 \frac{1}{r_i} = -4\pi \, \delta(x - x_i) \, \delta(y - y_i) \, \delta(z - z_i).$$

Such a limit , i.e., $R_i \to 0$ is not consistent in our case , $\nabla^2 \lambda + 4\pi \, \rho \, \lambda^3 = 0$. The potential equation is nonlinear and in particular in this limit the product of ρ and λ^3 does not make sense. Hence we remark that the Majumdar-Papapetrou metrics should represent the gravitational field N objects with nonempty interiors (not point-like objects).

5 Thin shell solutions

In the previous section we concluded that the dust distribution can not be concentrated to a point. We observe that , the potential equation (8) does not also admit dust distributions on one dimensional (string like distributions) structures. This is compatible with the results of Geroch-Traschen [9]. On the other hand , the mass distribution ρ can be defined on surfaces.

Let S be a regular surface in space (R^3) defined by $S = [(x, y, z) \in R^3; F(x, y, z) = 0]$, where F is a differentiable function in R^3. When the dust distribution is concentrated on S the mass density may be represented by the Dirac delta function

$$\rho(x, y, z) = \rho_0(x, y, z) \, \delta(F(x, y, z)) \tag{21}$$

where $\rho_0(x, y, z)$ is a function of (x, y, z) which is defined on S. The function λ satisfying the potential equation (8) compatible with such shell like distributions may given as

$$\lambda(x, y, z) = \lambda_0(x, y, z) - \lambda_1(x, y, z) \, \theta(F) \tag{22}$$

where λ_0 and λ_1 are differentiable functions of (x, y, z) and $\theta(F)$ is the Heaviside step function. With these assumptions we obtain

$$\rho_0(x,y,z) = \frac{1}{4\pi}\,\frac{\vec{\nabla}\lambda_1\cdot\vec{\nabla}F}{(\lambda_0)^3}|_S \tag{23}$$

$$\nabla^2\lambda_0 = \nabla^2\lambda_1 = 0 \tag{24}$$

and in addition $\lambda_1|_S = 0$. We have some examples:

1. S is the plane $z = 0$. We have $\rho(x,y,z) = \rho_0(x,y)\,\delta(z)$. Then it follows that

$$\lambda(x,y,z) = \lambda_0(x,y,z) - \lambda_2(x,y)\,z\,\theta(z) \tag{25}$$

$$\rho_0(x,y,z) = \frac{1}{4\pi}\,\frac{\lambda_2}{(\lambda_0|_S)^3} \tag{26}$$

$$\nabla^2\lambda_0 = \nabla^2\lambda_2 = 0 \tag{27}$$

2. S is the cylinder $F = r - a = 0$. We have $\rho(r,\theta,z) = \rho_0(\theta,z)\,\delta(r-a)$. Then it follows that

$$\lambda(r,\theta,z) = \lambda_0(r,\theta,z) - \lambda_2(\theta,z)\,\ln(r/a)\,\theta(\rho - a) \tag{28}$$

$$\rho_0(r,\theta,z) = \frac{1}{4\pi a}\,\frac{\lambda_2}{(\lambda_0|_S)^3} \tag{29}$$

$$\nabla^2\lambda_0 = \nabla^2\lambda_2 = 0 \tag{30}$$

Here we remark that the limit $a \to 0$ does not exist. This means that the mass distribution on the whole z- axes is not allowed.

3. S is the sphere $F = r - a = 0$. We have $\rho(r,\theta,\phi) = \rho_0(\theta,\phi)\,\delta(r-a)$. Then it follows that

$$\lambda(r,\theta,\phi) = \lambda_0(r,\theta,\phi) - \lambda_2(\theta,\phi)\,(\frac{1}{a} - \frac{1}{r})\,\theta(r-a) \tag{31}$$

$$\rho_0(r,\theta,\phi) = \frac{1}{4\pi a^2}\,\frac{\lambda_2}{(\lambda_0|_S)^3} \tag{32}$$

$$\nabla^2\lambda_0 = \nabla^2\lambda_2 = 0 \tag{33}$$

We note that the total mass is infinite on non-compact surfaces.

For compact case we shall consider the sphere in more detail. In this case we may have $\lambda_0 = \mu\,\lambda_2 + \psi$ such that $\nabla^2 \psi = 0$ and $\psi(a,\theta,\phi) = 0$. We shall assume $\psi = 0$ everywhere, then

$$\rho_0 = \frac{1}{4\pi a^2}\,\frac{1}{\mu^3\,\lambda_2^2} \tag{34}$$

Hence the total mass m_0 on S is given by $m_0 = \int \sqrt{-g}\,\rho\, d^3 x = \frac{1}{\mu}$. Let λ^{out} and λ^{in} denote solutions of (8) corresponding to the exterior and inner regions respectively. They are given by

$$\begin{aligned} \lambda^{out}(r,\theta,\phi) &= \lambda(r > a,\theta,\phi) = 1 - \frac{m_0}{a} + \frac{m_0}{r} && (35)\\ \lambda^{in}(r,\theta,\phi) &= \lambda(r < a,\theta,\phi) = 1, && (36) \end{aligned}$$

where we let $\lambda_2 = m_0$. Here we remark that the point particle limit $a \to 0$ does not exist. The solution given above represents the extreme Reissner-Nordström solution

$$ds^2 = -\,(1 - \frac{m_0}{a} + \frac{m_0}{r})^{-2} dt^2 + (1 - \frac{m_0}{a} + \frac{m_0}{r})^2\,(dr^2 + r^2\,d\,\Omega^2) \tag{37}$$

By letting $r = \frac{R - m_0}{\beta}$ where $\beta = 1 - \frac{m_0}{a}$ and $a \neq m_0$ is assumed. We obtain the extreme RN in its usual form

$$ds^2 = -\,\beta^2\,(1 - \frac{m_0}{R})^2 dt^2 + \frac{dR^2}{(1 - \frac{m_0}{R})^2} + R^2\,d\,\Omega^2 \tag{38}$$

Hence we obtain a solution where the exterior solution is the extreme Reissner - Nordström metric, but inside the sphere with radius a, the spacetime is flat. Thus extreme RN solution is matched to a spherical shell ($R = a$) of dust distrubition.

The case $a = m_0$ represents the Levi-Civita -Bertotti-Robinson spacetime outside the dust shell and the flat spacetime inside.

$$ds^2 = -\frac{r^2}{m_0^2}\,dt^2 + \frac{m_0^2}{r^2}\,dr^2 + m_0^2\,d\,\Omega^2 \tag{39}$$

By letting $r = \frac{m_0^2}{R}$ we obtain the usual conformally flat $LCBR$ metric

$$ds^2 = \frac{m_0^2}{R^2} \, [-dt^2 + dR^2 + R^2 \, d\Omega^2] \tag{40}$$

In the new coordinates the surface is again $R = m_0$. In these dust shell solutions the function λ is continuous on the surface $r = a$, but its normal derivative to S is discontinuous ,

$$\lambda'_{out} - \lambda'_{in} = -4\pi \, \sigma = -m_0/a^2$$

as expected, where $\sigma =$ mass per unit area $= m_0/4\pi a^2$. For thin shells in general relativity see the recent work of Mansouri and Khorrami [10] and also Mansouri's contribution in this proceeding.

6 Conclusion

We have solved the Einstein field equations in a conformo-static space-time for a charged dust distribution. We reduced the whole Einstein field equations to a nonlinear Poisson type of potential equation (8). Physically reasonable solutions of this equation give an interior solution to an exterior MP metrics. We have given some explicit exact solutions corresponding to some mass densities. In particular we have given an interior solution of the extreme RN metric.

We showed that the limiting cases of mass distributions on discrete points and also on lines in R^3 are not possible. We have examined some possible mass distributions on regular surfaces. We have found solutions corresponding to shell like dust distributions. In particular for the spherical dust shell we presented an exact solution given in eqns. (35-39) representing a spherical shell cavity immersed in the extreme RN spacetime.

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